Numerical evolution of the resistive relativistic MHD equations: a minimally implicit Runge-Kutta scheme

Mathematical and Computational Approaches for the Einstein Field Equations with Matter Fields

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Outline:

- **Motivation and structure** of the relativistic resistive magnetohydrodynamic equations.

- **IMEX** Runge-Kutta methods: high computational cost.

- Derivation of the **new schemes**: first and second-order methods.

- First **numerical simulations**.

- **Conclusions and future plans**.
Motivation and structure of the (special) relativistic resistive magnetohydrodynamic equations.

Motivations for considering the non-ideal magnetohydrodynamic (MHD) equations (see A. Christlieb's talk yesterday):

- Significant magnetic field in some astrophysical scenarios: active galactic nuclei, quasars, compact objects, dolls relativistes, accretion disks...

- Numerical simulations in the ideal case: effects coming from the numerical error and numerical resistivity (dependence on the numerical method and resolutions used), physical resistivity is not modeled consistently.

- High resolution shock capturing methods for capturing shock waves and rarefaction waves.

- Hyperbolic evolution equations + constraint equations (zero divergence of magnetic field).
Motivation and structure of the (special) relativistic resistive magnetohydrodynamic equations.

\[
\begin{align*}
\nabla \cdot B &= 0 \\
\partial_t B + \nabla \times E &= 0
\end{align*}
\]

\[
\begin{align*}
\partial_t \phi + \nabla \cdot B &= - k \phi \\
\partial_t B + \nabla \times E + \nabla \phi &= 0
\end{align*}
\]

\[
\begin{align*}
\nabla \cdot E &= q \\
- \partial_t E + \nabla \times B &= J
\end{align*}
\]

\[
\begin{align*}
\partial_t \psi + \nabla \cdot E &= q - k \psi \\
- \partial_t E + \nabla \times B - \nabla \psi &= J
\end{align*}
\]

Constraint violations decay exponentially and propagate at speed of light.

Augmented evolution system for the new set of conserved variables [Komissarov, 2007].

\[
J^i = \sigma W (E^i + (v \times B)^i - E_j v^j v^i) + q v^i
\]

\[
e = (E^2 + B^2)/2 + \rho h W^2 - p
\]

\[
P^i = S^i = (E \times B)^i + \rho h W^2 v^i
\]
Motivation and structure of the (special) relativistic resistive magnetohydrodynamic equations.

- Conserved and primitive variables (geometric units):

\[ x^A = \{\phi, B^i, \psi, E^i, q, (\rho W), e, P^i\} \]

\[ x^{A'} = \{\phi, B^i, \psi, E^i, q, \rho, e, v^i\} \]

The Lorentz factor is defined in terms of primitive variables:

\[ W = (1 - v^2)^{-1/2} \]

- System of equations:

\[ \partial_t E^j = S^j_E - \sigma W \left[ E^j + (v \times B)^j - E^l v_l v^j \right] \]

\[ \partial_t Y = S_Y \]

\[ \partial_t B^j = S_B^j \]

Conductivity times Lorentz factor: potential stiff source term
**IMEX Runge-Kutta methods**

The presence of *stiff source terms* needs an *implicit* treatment of the source term or part of the source term.

A hyperbolic equation with a *relaxation term* has the form:

\[
\partial_t U = F(U) + \frac{1}{\epsilon} R(U)
\]

\(R(U)\) has no derivatives with respect to the variable \(U\) (source term). Potential stiff source term for \(\Delta t \leq \epsilon\).

Previously used *methods*:
- Strang-splitting method.
- [Palenzuela, Lehner, Reula, Rezzolla (2009)] IMEX Runge-Kutta methods:

\[
\partial_t Y = F_Y(X, Y)
\]

\[
\partial_t X = F_X(X, Y) + \frac{1}{\epsilon(Y)} R_X(X, Y)
\]

\[
R_X(X, Y) = A(Y)X + S_X(Y)
\]
IMEX Runge-Kutta methods

[Palenzuela, Lehner, Reula, Rezzolla (2009)] IMEX Runge-Kutta methods:

- Successfully used in several numerical experiments: Afvén waves with high amplitude and high conductivity to get similar results with respect to the ideal case; broad range of values for the conductivity in shock tubes; neutron star with magnetic field.

- The implicit part involves the Lorentz factor, defined in terms of primitive variables (components of the velocity field).

- Computationally expensive: reconstruction of variables implemented in each time-step, nested iterative loops for recovery of primitive variables without guarantee of convergence.
Alternative approach: minimally-implicit Runge-Kutta methods

\[
\begin{align*}
\partial_t E^j &= S_E^j - \sigma W \left[ E^j + (v \times B)^j - E^l v_l v^j \right] \\
\partial_t Y &= S_Y \\
\partial_t B^j &= S_B^j 
\end{align*}
\]

Implicit terms

First-order method:

\[
\begin{align*}
E^j_{n+1} &= E^j_n + \Delta t \left[ S_E^j_n - \sigma W_n \left[ c_1 E^j_n + (1 - c_1) E^j_{n+1} \\
+ c_2 \ (v \times B)^j_n + (1 - c_2) \ (v_n \times B_{n+1})^j \\
- c_3 \ v^j_n v_l |_n E^l |_n - (1 - c_3) \ v^j_n v_l |_n E^l |_{n+1} \right] \right] \\
Y_{n+1} &= Y_n + \Delta t S_Y_n \\
B^j_{n+1} &= B^j_n + \Delta t S_B^j_n,
\end{align*}
\]

Effective conductivity:

\[
\overline{\sigma} = \Delta t \left( \sigma W \right)_n
\]
Alternative approach: minimally-implicit Runge-Kutta methods

Stability analysis based on:

· **Finite values** for very high values of the effective conductivity.

\[(1 - c_1) \neq 0 \quad (1 - c_1 + v^2|n(c_3 - 1)) \neq 0\]

· Recovery of **ideal limit**.

· Wave-like behaviour between magnetic and electric fields → recovery of PIRK method for explicit part.

\[c_2 = 0\]

· **Linear stability analysis** for infinite conductivity: additional simplification + one eigenvalue set to zero for any velocity (dependence of electric field on the rest of the variables).

\[c_3 = 1 \quad c_1 = 0\]

· The other eigenvalue is bounded by 1 in absolute value for any velocity.
Alternative approach: 
minimally-implicit Runge-Kutta methods

First-order method:

\[ E^i|_{n+1} = E^i|_n + \frac{1}{1 + \bar{\sigma}} \left\{ \Delta t \, S_E^i|_n + \bar{\sigma} \, E^l|_n \right\} \left[ -\delta^i_l + v^i|_n \, v_l|_n \right] - \bar{\sigma} \left( v|_n \times B|_{n+1}^i \right) \]

Explicit scheme with an effective time-step: \( \Delta t/(1 + \bar{\sigma}) \)


\[
\begin{align*}
E^j|_{(1)} &= E^j|_n + \Delta t \, S_E^j|_n - \bar{\sigma} \left[ c_1 \, E^j|_n + (1 - c_1) \, E^j|_{(1)} \right] \\
&\quad - \bar{\sigma} \left[ c_2 \, (v \times B)^j|_n + (1 - c_2) \, (v|_n \times B|_{(1)})^j \right] \\
&\quad + \bar{\sigma} \, v^j|_n \, v_l|_n \left[ c_3 \, E^l|_n + (1 - c_3) \, E^l|_{(1)} \right], \\
Y|_{(1)} &= Y|_n + \Delta t \, S_Y|_n, \\
B|_{(1)} &= B|_n + \Delta t \, S_B^j|_n,
\end{align*}
\]
Alternative approach: 
minimally-implicit Runge-Kutta methods


\[
E^j |_{n+1} = \frac{1}{2} [E^j |_n + E^j |_{(1)} + \Delta t S^j_E |_{(1)}] \\
-\bar{\sigma} \left[ \frac{(1 - c_1)}{2} E^j |_n + c_4 E^j |_{(1)} + (c_1/2 - c_4) E^j |_{n+1} \right] \\
-\bar{\sigma} \left[ \frac{(1 - c_2)}{2} (v |_{(1)} \times B |_n)^j + c_5 (v \times B)^j |_{(1)} + (c_2/2 - c_5) (v |_{(1)} \times B |_{n+1})^j \right] \\
+\bar{\sigma} v^j |_{(1)} v^l |_{(1)} \left[ \frac{(1 - c_3)}{2} E^l |_n + c_6 E^l |_{(1)} + \left( \frac{c_3}{2} - c_6 \right) E^l |_{n+1} \right]
\]

\[
Y |_{n+1} = \frac{1}{2} [Y |_n + Y |_{(1)} + \Delta t S_Y |_{(1)}] \\
B^j |_{n+1} = \frac{1}{2} [B^j |_n + B^j |_{(1)} + \Delta t S^j_B |_{(1)}]
\]

\[
\bar{\sigma} = \Delta t \sigma W |_n, \quad \bar{\sigma} = \Delta t \sigma W |_{(1)}
\]
Alternative approach: minimally-implicit Runge-Kutta methods

Stability analysis based on the same previous points:

· **Finite values** for very high values of the effective conductivity.

\[
(1 - c_1) \neq 0; \quad (1 - c_1 + v^2|_n(c_3 - 1)) \neq 0; \\
(c_1/2 - c_4) \neq 0; \quad (c_1/2 - c_4 - v^2|_n(c_3/2 - c_6)) \neq 0.
\]

· Recovery of **ideal limit**.

· Recovery of **PIRK method** for explicit part.

\[
c_2 = 1 - \frac{\sqrt{2}}{2}, \quad c_5 = \frac{\sqrt{2} - 1}{2}
\]
Alternative approach: minimally-implicit Runge-Kutta methods

- Linear stability analysis for infinite conductivity:

(i) additional simplification.

\[ c_3 = 1, \ c_6 = 1/2 \]

(ii) one eigenvalue set to zero.

\[ c_1 \neq 0, \quad c_4 = \frac{(1 - c_1)^2}{2c_1} \]

(iii) the second eigenvalue bounded by 1 in absolute value for any velocity in a stable way.

\[ c_1 < 0 \]

(iv) the second eigenvalue is minimum with respect to the remaining coefficient.

\[ c_1 = -\frac{1}{\sqrt{2}} \]
First numerical simulations

Evolution of magnetic and electric field. 
Charge computed from divergence of electric field.

Finite differences, equally spaced grid and cartesian coordinates. 
CFL factor = 0.8

Constant velocity components and conductivity.

Set up for initial data:

\[
\begin{align*}
B &= (0, B^y(x,t), 0) \\
E &= (0, 0, 0) \\
v &= (v^x, v^y, 0) \\
\phi &= 0
\end{align*}
\]

\[
B^y(x,t) \cdot E^z(x,t)
\]

\[
B^x(x, t = 0) = erf \left( \frac{x \sqrt{\sigma}}{2} \right)
\]

Can be chosen to be zero
Both explicit and implicit methods works fine if conductivity is not very high, resolution is not very small or velocity is zero.

\[
\sigma = 100, \quad \Delta x = 0.015, \quad v_x = 0
\]
\[ \Delta x = 0.015, \, \sigma = 1000, \, v_x = 0.1 \]
Conclusions:

- **Simple** first and second order schemes, minimizing the implicit parts. Only **conserved variables** are included in these terms. Analytical trivial inversion of the operators.

- **Stability** conditions close to **ideal limit** are used to select values for the coefficients. **No need of iterative** schemes on each stage (apart from recovery), **effective time-step**.

- First **numerical simulations**. Future more complex ones.

- Comparison with other approaches: well-balanced methods.

Thanks for your attention... next time hopefully more movies!!